Analytical solutions for steady state vertical infiltration

Jianting Zhu and Binayak P. Mohanty

Biological and Agricultural Engineering Department, Texas A&M University, College Station, Texas, USA

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[1] In this note, some analytical solutions for steady state vertical infiltration are presented. The forms of unsaturated hydraulic conductivity function used in the study are the rational power model of the form $K(\psi) = K_S/[1 + (\alpha\psi)^\beta]$ and the *Brooks and Corey* [1964] model. The work presented in this note complements *Warrick*'s [1988] solutions for evaporation and provides analytical solutions for vertical infiltration. *INDEX TERMS:* 1875 Hydrology: Unsaturated zone; 1866 Hydrology: Soil moisture; 1836 Hydrology: Hydrologic budget (1655); 1829 Hydrology: Groundwater hydrology; *KEYWORDS:* Steady state infiltration, unsaturated hydraulic conductivity, rational power model, Brooks and Corey model

1. Introduction

[2] One-dimensional steady state vertical flow is a practical assumption in many soil hydrologic studies because the gradients would be relatively very small in the horizontal direction. The one-dimensional Richards' equation was solved by *Warrick* [1988] to consider evaporation from a shallow water table. Two types of unsaturated hydraulic conductivity functions were used in his study, i.e., the rational power model of *Gardner*'s [1958] type with all real power values and the *Brooks and Corey* [1964] model. *Warrick*'s [1988] solutions are not suitable for the case of infiltration (i.e., downward flux q). The analytical solutions were derived by *Warrick* [1991] and *Gardner* [1958] for the relatively simple exponential hydraulic conductivity function of *Gardner* [1958]. To the best of our knowledge, there are no analytical solutions available in the literature for the infiltration case. There are only a few approximate solutions available for the problem. Among them, *Salvucci* [1993] obtained an approximate solution with explicit expressions for capillary tension as a function of depth for both evaporation and infiltration using the rational power model. More recently, *Basha* [1999] presented an approximate solution of the steady infiltration with arbitrary plant withdrawal for the rational power model. We now extend analytical solutions to the infiltration case, and this work serves to fill in the existing gap in terms of analytical solution.

2. One-Dimensional Steady State Infiltration

[3] The capillary tension profile can be determined by the Darcy's equation where one-dimensional steady state infiltration can be expressed as

$$-q = K(\psi)\left(\frac{d\psi}{dz} - 1\right), \tag{1}$$

where q is the infiltration rate, K is the unsaturated hydraulic conductivity, ψ is the capillary tension, and z is the vertical distance (positive upward) with $z = 0$ at the water table where $\psi = 0$. Then equation (1) leads to

$$z = \int_0^\psi \frac{K(\psi)d\psi}{K(\psi) - q}. \tag{2}$$

2.1. Rational Power Model

[4] Following *Gardner* [1958], we employ the following unsaturated hydraulic conductivity model to relate the capillary tension to the reduction of hydraulic conductivity from its saturated value K_S:

$$K(\psi) = \frac{K_S}{1 + (\alpha\psi)^\beta}. \tag{3}$$

Substitution of equation (3) to equation (2) leads to

$$z = \int_0^\psi \frac{K(\psi)d\psi}{K(\psi) - q} = \frac{1}{1 - q'}\int_0^\psi \frac{d\psi}{1 - q'(\alpha\psi)^\beta/(1 - q')}, \tag{4}$$

where q' is the normalized infiltration flux, $q' = q/K_S$.

[5] After the introduction of a new variable,

$$v = \left[1 - \frac{q'(\alpha\psi)^\beta}{1 - q'}\right]^{-1} - 1,$$

that is,

$$d\psi = \frac{\alpha}{\beta}\left(\frac{q'}{1 - q'}\right)^{1/\beta} v^{1/\beta - 1}(1 + v)^{-1-1/\beta}dv,$$

we can write the integration as follows:

$$z = \frac{\alpha q'^{1/\beta}}{\beta(1 - q')^{(\beta+1)/\beta}}\int_0^v \frac{v^{1/\beta-1}dv}{(1 + v)^{1/\beta}}. \tag{5}$$

It can be integrated further [cf. *Gradshteyn and Ryzhik*, 1994, equation (3.194), p. 333] to

$$z = \frac{\alpha q'^{1/\beta}}{(1 - q')^{(\beta+1)/\beta}} v^{1/\beta}\, {}_2F_1(1/\beta, 1/\beta; 1 + 1/\beta; -v)$$
$$= \frac{\alpha q'^{1/\beta}}{(1 - q')^{(\beta+1)/\beta}}\left(\frac{v}{v + 1}\right)^{1/\beta} {}_2F_1\left(1/\beta, 1; 1 + 1/\beta; \frac{v}{v + 1}\right), \tag{6}$$

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where ${}_2F_1$ is the hypergeometric function. The function may be expressed in a series form, and the solution can be written in the following form:

$$z = \frac{\alpha q'^{1/\beta}}{(1-q')^{(\beta+1)/\beta}} \left(\frac{v}{v+1}\right)^{1/\beta} \sum_{j=0}^{\infty} b_j, \tag{7}$$

where the terms in the series can be calculated in the following recursive way:

$$b_0 = 1 \tag{8a}$$

$$b_{j+1} = \frac{(j+1/\beta) v b_j}{(j+1+1/\beta)(v+1)} \qquad j \geq 0. \tag{8b}$$

2.2. Brooks and Corey Model

[6] The Brooks and Corey model has the following form:

$$K(\psi) = \frac{K_S}{(\alpha\psi)^{\lambda(\ell+2)+2}} = K_s(\alpha\psi)^{-\beta} \qquad \alpha\psi > 1 \tag{9a}$$

$$K(\psi) = K_s \qquad \alpha\psi \leq 1, \tag{9b}$$

where, $\beta = \lambda(\ell + 2) + 2$, λ is the pore size distribution parameter in the Brooks and Corey retention function, and ℓ is the pore connectivity parameter in the relative hydraulic conductivity model proposed by *Mualem* [1976].

[7] If $\alpha\,\psi \leq 1$, the relationship between ψ and z is simply

$$z = \frac{K_S \psi}{K_S - q} = \frac{\psi}{1-q'}.$$

If $\alpha\psi > 1$, then the integration in equation (2) can be written as

$$\begin{aligned} z &= \int_0^{\psi} \frac{K(\psi)d\psi}{K(\psi)-q} = \int_0^{1/\alpha} \frac{d\psi}{1-q'} + \int_{1/\alpha}^{\psi} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} \\ &= \int_0^{1/\alpha} \frac{d\psi}{1-q'} + \int_0^{\psi} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} - \int_0^{1/\alpha} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} \\ &= \int_0^{\psi} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} + \int_0^{1/\alpha} \left[\frac{d\psi}{1-q'} - \frac{d\psi}{1-q'(\alpha\psi)^{\beta}}\right] \\ &= \int_0^{\psi} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} + \frac{q'}{1-q'} \int_0^{1/\alpha} \frac{1-(\alpha\psi)^{\beta}}{1-q'(\alpha\psi)^{\beta}} d\psi. \end{aligned} \tag{10}$$

The first term in equation (10) can be carried out in a similar way with rational power model case, which can be written as

$$\int_0^{\psi} \frac{d\psi}{1-q'(\alpha\psi)^{\beta}} = \frac{1}{\alpha}\left[\frac{s}{q'(1+s)}\right]^{1/\beta} {}_2F_1\left(\frac{1}{\beta}, 1; 1+\frac{1}{\beta}; \frac{s}{1+s}\right), \tag{11}$$

where $s = [1 - q'(\alpha\psi)^{\beta}]^{-1} - 1$. The second integration in equation (10) can be carried out with a change of variable, $x = (\alpha\psi)^{\beta}$. It leads to

$$\int_0^{1/\alpha} \frac{1-(\alpha\psi)^{\beta}}{1-q'(\alpha\psi)^{\beta}} d\psi = \frac{1}{\alpha\beta} \int_0^1 x^{1/\beta-1}(1-x)(1-q'x)^{-1} dx. \tag{12}$$

Table 1. Representative Values for the Brooks and Corey and the Rational Power Model Soil Parameters and Normalized Flow Rate

	K_S, cm s^{-1}	α, cm^{-1}	β	q'
Clay	3.4×10^{-5}	0.0111	3.3	0.1
Silt-loam	3.4×10^{-4}	0.0222	5.64	0.01
Sand-loam	3.4×10^{-3}	0.0400	11.88	0.001

The integration can be carried out in terms of the β function and the hypergeometric function [cf. *Gradshteyn and Ryzhik*, 1994, equation (3.197), p. 335]:

$$\begin{aligned} \int_0^{1/\alpha} \frac{1-(\alpha\psi)^{\beta}}{1-q'(\alpha\psi)^{\beta}} d\psi &= \frac{1}{\alpha\beta} B(1/\beta, 2) \;\; {}_2F_1(1, 1/\beta; 2+1/\beta; q') \\ &= \frac{\beta}{\alpha(1+\beta)} {}_2F_1(1, 1/\beta; 2+1/\beta; q'). \end{aligned} \tag{13}$$

Therefore the combination of these two terms results in the following complete solution for Brooks and Corey model:

$$z = \psi/(1-q') \qquad \alpha\psi \leq 1 \tag{14a}$$

$$z = \frac{\beta q'}{\alpha(1-q')(1+\beta)} \sum_{j=0}^{\infty} e_j + \psi \sum_{j=0}^{\infty} f_j \qquad \alpha\psi > 1 \tag{14b}$$

with

$$e_0 = 1, \tag{15a}$$

$$e_{j+1} = \frac{(j+1/\beta) q' e_j}{(j+2+1/\beta)} \qquad j \geq 0, \tag{15b}$$

$$f_0 = 1, \tag{16a}$$

$$f_{j+1} = \frac{(j+1/\beta) s f_j}{(j+1+1/\beta)(1+s)} \qquad j \geq 0. \tag{16b}$$

We would like to point out a probable typo in *Warrick*'s [1988, p. 64] paper. The second part of his equation (25) should read as

$$a_{j+1} = \frac{(2+j)\gamma a_j}{[2+j+(1/n)]} \qquad j \geq 0.$$

[8] The solutions presented in this paper can be easily extended to the case where the capillary tension (ψ_L) is given at the lower boundary (z_L) by the following simple manipulation:

$$z - z_L = \int_{\psi_L}^{\psi} \frac{K(\psi)d\psi}{K(\psi)-q} = \int_0^{\psi} \frac{K(\psi)d\psi}{K(\psi)-q} - \int_0^{\psi_L} \frac{K(\psi)d\psi}{K(\psi)-q}. \tag{17}$$

The results we developed earlier can then be used to evaluate each term in the equation (17).

3. Sample Results

[9] In the dry range the rational power model closely mimics the Brooks and Corey model. In the wet range the

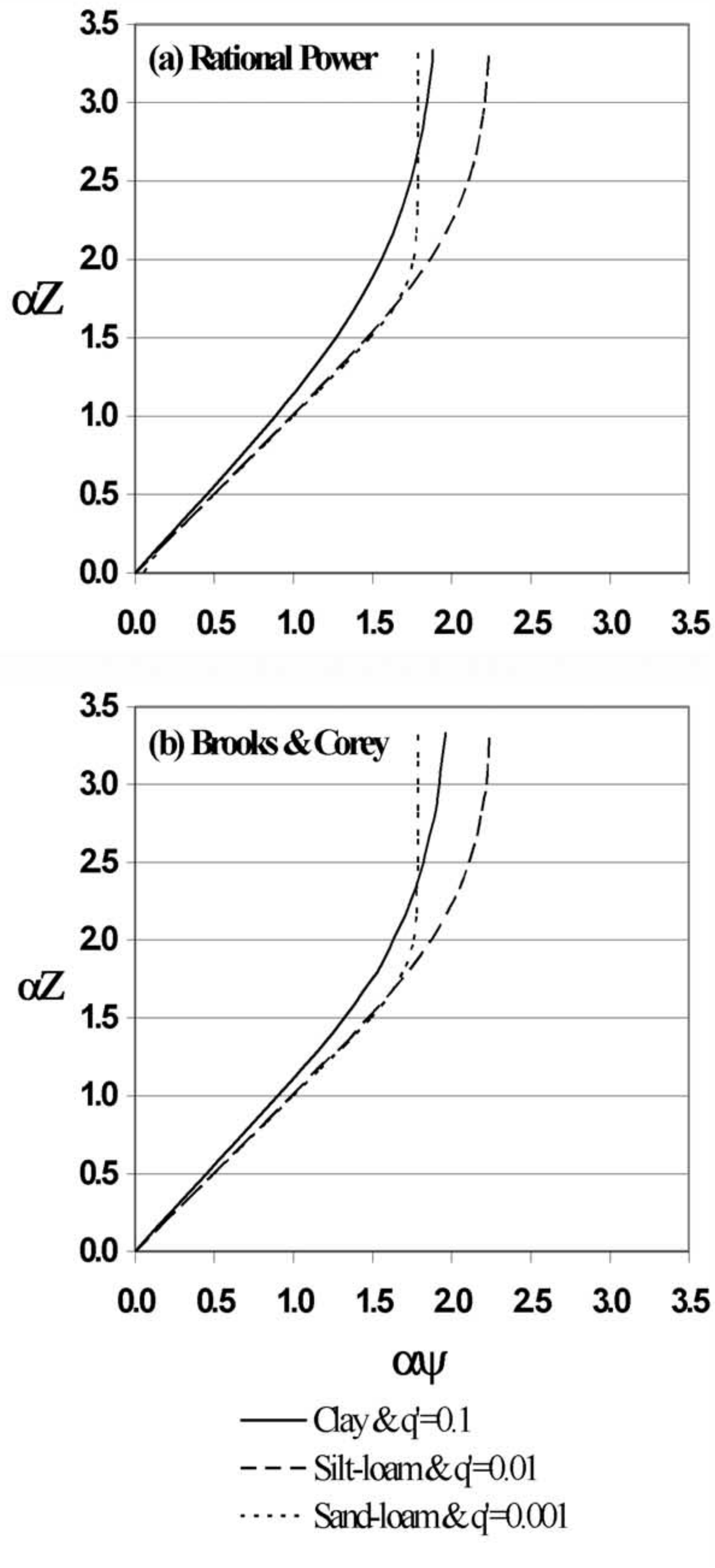

Figure 1. Capillary tension profiles versus distance above water table. (a) For the rational power model. (b) For the Brooks and Corey model.

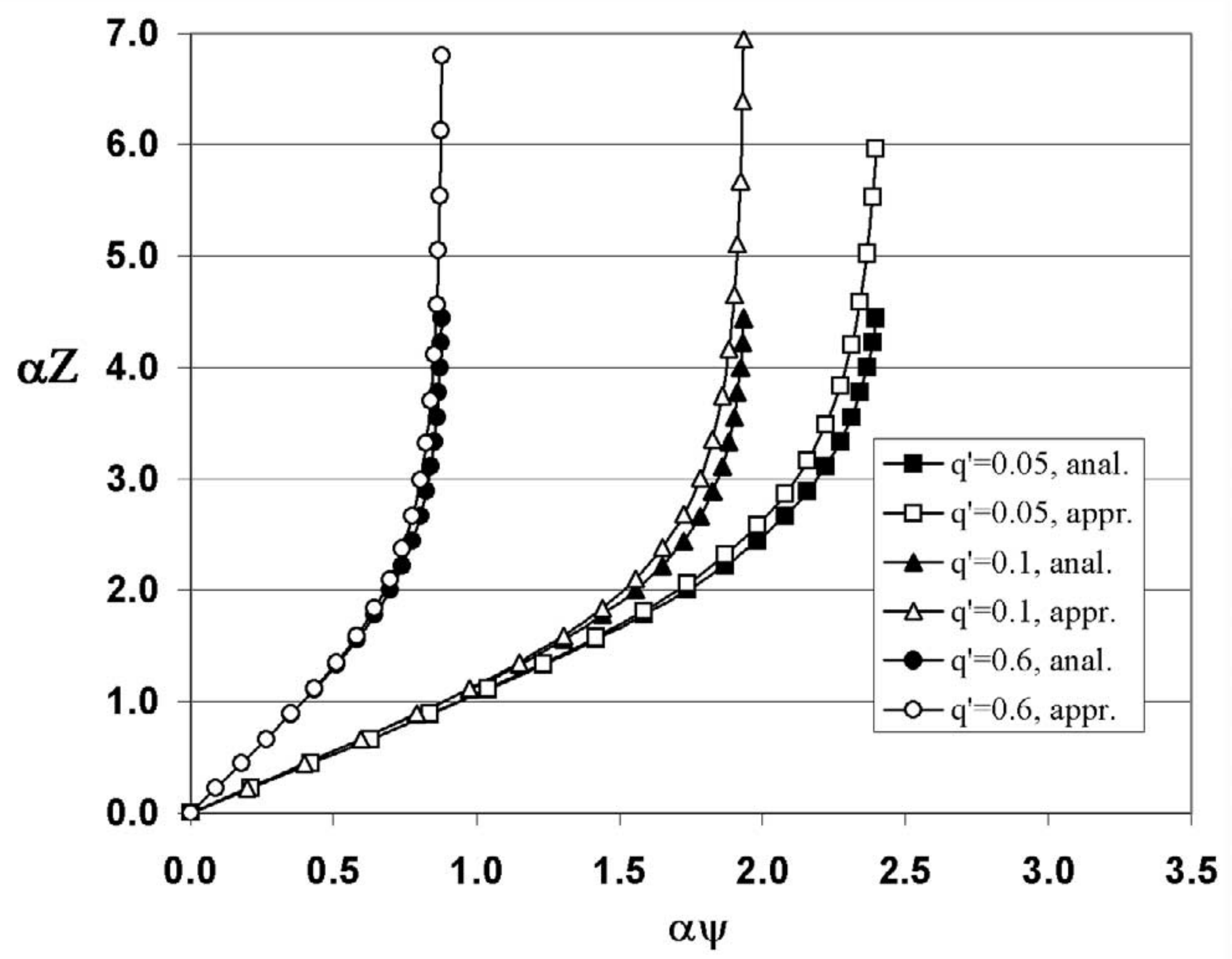

Figure 2. Comparison between the results of this study and *Salvucci* [1993] for the clay soil case at three different infiltration rates.

rational power model smoothes out the discontinuity present in the Brooks and Corey model. In the sample calculations we choose the input values for K_S, α, β, and q' from *Salvucci* [1993] and *Bras* [1990] for three soil textures. They are listed in Table 1.

[10] The capillary tension profiles for the three soil textures are shown in Figure 1a for the rational power model and in Figure 1b for the Brooks and Corey model. The results for the two models compare quite similarly except for the clay case where the capillary tension for the Brooks and Corey model is slightly larger than that for the rational power model. This is because the rational power model noticeably smoothes out the discontinuity existing in the Brooks and Corey model, resulting in larger differences between these two models for clayey soil than those for sand-loam and silt-loam soils.

[11] *Salvucci* [1993] developed an approximate solution for the rational power model by expanding and truncating the integrand in the analytical expression as follows:

$$\alpha z = \frac{1}{1-q'}\left[(\alpha\psi)^{-\beta} - \frac{q'}{1-q'}\right]^{-1/\beta}. \qquad (18)$$

In Figure 2 we show comparison between our results and Salvucci's results for the clay case at three different infiltration rates. It is apparent that Salvucci's approximate solutions consistently overestimated the depth for the steady state infiltration. The way this solution was derived contributes to the fact that it overestimated the depth z. In *Salvucci*'s [1993, equation (12)] solution the term $(y+I)^{-1/\beta}$ was approximated by $y^{-1/\beta}$ in the integrand to obtain z, which was obviously an overestimation. As we can see, the overestimation could be fairly significant for the fine-textured soil (clay).

4. Concluding Remarks

[12] For soils with their unsaturated hydraulic conductivity characterized by *Gardner*'s [1958] rational power model or the Brooks and Corey model, we developed analytical solutions for the steady state one-dimensional infiltration in terms of the relationships between the capillary tension and the distance above the water table. These solutions complement *Warrick*'s [1988] solutions for evaporation and may be used either for calculations of capillary tension profiles or for checking numerical computations.

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B. P. Mohanty and J. Zhu, Biological and Agricultural Engineering Department, Scoates Hall, Texas A&M University, College Station, TX 77843-2117, USA. (bmohanty@tamu.edu; jzhu@cora.tamu.edu)